P.4: Complex Numbers

4/1) Def 1: The imaginary number i is defined by

$$i^2 = -1 \quad \text{-or-} \quad i = \sqrt{-1}.$$

Def 2: The set of complex numbers, denoted $\mathbb{C}$, is the set of all numbers of the form $a+bi$ where a and b are real numbers.

The real part of $a+bi$ is a and the imaginary part is b. Two complex numbers $a+bi$ and $c+di$ are equal if and only if $a=c$ and $b=d$. $a+bi$ is called the standard form of a complex number. For example $3(2+i)$ is not is standard form, but $6+3i$ is.

Just as with real numbers, we can add, subtract, multiply, and divide complex numbers

Def 3: Let $a+bi$ and $c+di$ be complex numbers. Then

1) $(a+bi)+(c+di) = (a+c)+(b+d)i$
2) $(a+bi)-(c+di) = (a-c)+(b-d)i$
3) $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$

(since $(a+bi)(c+di) = ac+bci+adi+bdi^2 = ac+bci+adi-bd$)

Ex 1: Perform the operation and rewrite in standard form.

a) $(3-3i)+(4+5i)$

b) $\left(\frac{1}{2}-\frac{2}{3}i\right)-\left(3-\frac{1}{4}i\right)$

c) $(4-5i)(2-3i)$

Sol: a) $(3-3i)+(4+5i) = (3+4)+(-3+5)i = \boxed{7+2i}$

b) $\left(\frac{1}{2}-\frac{2}{3}i\right)-\left(3-\frac{1}{4}i\right) = \left(\frac{1}{2}-3\right)+\left(\frac{-2}{3}-\frac{-1}{4}\right)i$

$= \left(\frac{1}{2}-\frac{6}{2}\right)+\left(\frac{-8}{12}+\frac{3}{12}\right)i = \boxed{\frac{-5}{2}-\frac{5}{12}i}$

c) $(4-5i)(2-3i) = (8-15)+(-12-10)i$

$= \boxed{-7-22i}$

SKIP SKIP

We can find any whole number power of i as follows:

First know that:

$i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i = i^1$

So we have a cycle of four things.

Suppose we have i^n. Divide n by 4 and let r be the remainder. Then $i^n = i^r$ as $i^n = i^{4q+r} = i^{4q}\cdot i^r = (i^4)^q\cdot i^r = 1^q\cdot i^r = i^r$

Ex 2: Find i^{35}.

$$\begin{array}{r} 8\ R3 \\ 4\overline{)35} \\ \underline{32} \\ 3 \end{array}$$

So $i^{35} = i^3 = \boxed{-i}$

Now we would like to define division of complex numbers, but first:

Def 4: The complex conjugate of the complex number $a+bi$ is $a-bi$.

Ex 3 Find the complex conjugate.
a) $3-9i$ b) $-5i$ c) 13

Sol: a) $\boxed{3+9i}$ b) $\boxed{5i}$ c) $\boxed{13}$

Ex 4: Find the product of the complex numbers in ex 3 with their conjugate.

Sol: a) $(3-9i)(3+9i) = 9+81 = \boxed{90}$
b) $(5i)(-5i) = \boxed{25}$
c) $(13)(13) = \boxed{169}$

Notice that all of the answers are positive. In fact:

$$(a+bi)(a-bi) = a^2+abi-abi+b^2 = a^2+b^2$$

Now we are ready to divide:

Def 5: To divide $a+bi$ by $c+di$ multiply $\frac{a+bi}{c+di}$ on the top and bottom by the conjugate of $c+di$.

In fact:

$$\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Ex 5: Divide: and write in standard form.

a) $\frac{1}{2-i}$ b) $\frac{-2-4i}{-i}$ c) $\frac{4+2i}{5-3i}$

Sol: a) $\frac{1}{2-i}\cdot\frac{2+i}{2+i}=\frac{2+i}{2^2+1^2}=\frac{2+i}{5}=\boxed{\frac{2}{5}+\frac{1}{5}i}$

b) $\frac{-2-4i}{-i}\cdot\frac{i}{i}=\frac{-2i-4i^2}{1}=\boxed{4-2i}$

c) $\frac{4+2i}{5-3i}\cdot\frac{5+3i}{5+3i}=\frac{(20-6)+(12+10)i}{5^2+3^2}=\frac{14+22i}{25+9}$

$=\frac{14+22i}{34}=\frac{14}{34}+\frac{22}{34}i=\boxed{\frac{7}{17}+\frac{11}{17}i}$

Def 6: For any $b>0$

$\sqrt{-b}=i\sqrt{b}$

P.5 : Polynomials

Adding and Subtracting Polynomials

Ex 1: a) $(-x^2+3x)+(x^2-5x+1)$

b) $(-3x^2-x+1)-(x^2-9)$

Sol: a) $(-x^2+3x)+(x^2-5x+1) = \boxed{-2x+1}$

b)
$$\begin{array}{r} -3x^2 - x + 1 \\ -\quad x^2 \qquad - 9 \\ \hline \boxed{-4x^2 - x + 10} \end{array}$$

Multiplying Polynomials

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$(a+b)(c+d)$

Ex 2: a) $(3x^2)(4x^5)$

b) $(2x+1)^2$

c) $(2x+3)(a+3)$

Sol: a) $(3x^2)(4x^5) = \boxed{12x^7}$

b) $(2x+1)^2 = 4x^2+2x+2x+1 = \boxed{4x^2+4x+1}$

c) $(2x+3)(a+3) = \boxed{2xa+6x+3a+9}$

Some Special Products

Prop 1:

a) $(a+b)^2 = a^2+2ab+b^2$

b) $(a-b)^2 = a^2-2ab+b^2$

c) $(a+b)(a-b) = a^2-b^2$

The degree of a polynomial is the highest exponent in the polynomial.

Dividing Polynomials

When we divide a polynomial $P(x)$ by $D(x)$ we get an equation

$$P(x) = Q(x)\,D(x) + R(x)$$

where $Q(x)$ is the quotient polynomial and $R(x)$ is the remainder polynomial. The degree of R must be less than the degree of D.

<u>Ex 3</u>: Divide: a) $\dfrac{x^2-3x+2}{x-1}$ b) $\dfrac{x^2-x-9}{x-3}$

<u>Sol</u>: a)

$$\begin{array}{r|l} & \quad x-2 \\ \hline x-1 & x^2-3x+2 \\ & -(x^2-x) \\ \hline & \quad -2x+2 \\ & -(-2x+2) \\ \hline & \qquad 0 \end{array}$$

So $\dfrac{x^2-3x+2}{x-1} = \boxed{x-2}$

b)

$$\begin{array}{r|l} & \quad x+2 \\ \hline x-3 & x^2-x-9 \\ & -(x^2-3x) \\ \hline & \quad 2x-9 \\ & -(2x-6) \\ \hline & \quad -3 \end{array}$$

So $\dfrac{x^2-x-9}{x-3} = \boxed{x+2-\dfrac{3}{x-3}}$

Ex 4: Using Conjugate to Rationalize a Denominator.

a) $\frac{2}{2-\sqrt{2}}$ b) $\frac{2}{3+\sqrt{5}}$

Sol:

a) $\frac{2}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{4+2\sqrt{2}}{2^2-(\sqrt{2})^2} = \frac{4+2\sqrt{2}}{4-2} = \frac{4+2\sqrt{2}}{2}$

$= \boxed{2+\sqrt{2}}$

b) $\frac{2}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{6-2\sqrt{5}}{3^2-(\sqrt{5})^2} = \frac{6-2\sqrt{5}}{9-5} = \frac{6-2\sqrt{5}}{4}$

$= \boxed{\frac{3-\sqrt{5}}{2}}$